Models for Global Plasma Dynamics

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Outline

1. Models for Long-Wavelength Plasma Dynamics
   - Introduction
   - Fluid models: MHD and Hall-MHD
   - Gyrokinetic theory
2. Hamiltonian Gyrofluid Reconnection
   - Motivation
   - HEMGF: A Hamiltonian Gyrofluid Model
   - Reconnection

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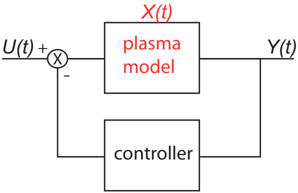
Relationship to control design process



$$\dot{X}(t) = f(X(t), U(t), t);$$
$$Y(t) = g(X(t), U(t), t).$$

- We focus on first two elements of control design process
  1. Make system model
  2. Verify model predicts behavior of system
  3. Design controller
  4. Test models in closed-loop simulation
  5. Implement and test implementation
  6. Deploy in operation
- We are concerned in this and subsequent talks with formulating and solving the equation for $X(t)$.

State space and Equilibrium

- Plasmas have infinite degrees of freedom, so X is a denumerably infinite-dimensional vector. We can think of it in terms of its Fourier coefficients. To represent X on a computer it must be *truncated*. Understanding the effect of truncation is part of the modeling task.
- Think of plasma as a system close to a quiescent, equilibrium state X_0:

$$\dot{X}_0 = F(X_0, U, t) = 0$$

Note that $X_0 = X_0(U)$.

Linear analysis of dynamics

- Small motions away from the equilibrium state, $X = X_0 + x$ can be described by Taylor expansion of the force:

$$\dot{x} = \mathbf{M}x + O(x^2),$$

where **M** is the Jacobian operator (matrix):

$$\mathbf{M} = \partial F(X, U, t)/\partial X|_{X=X_0} .$$

- Plasma models are usually *non-normal* ($\mathbf{M}^*\mathbf{M} \neq \mathbf{MM}^*$): they must be described by Singular Value Decomposition:

$$\mathbf{M} = \sum i\omega_j \, |v_j\rangle\langle u_j|,$$

MHD is an important exception where $\mathbf{M}^* = \mathbf{M}$.

Linear solution

- The singular value decomposition leads to the solution:

$$X(t) = X_0 + \sum e^{i\omega_j t} \, |v_j\rangle\langle u_j|.$$

- In practice, nonlinear effects are often important.
- To develop a useful model, we need to eliminate irrelevant time scales: for designing an steam engine, plate tectonics and nuclear vibrations are irrelevant.

Time scales I: The centrifugal governor

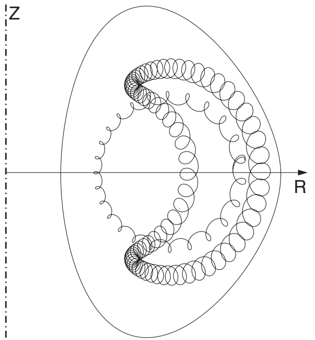
- T. Mead (1787), J.C. Maxwell (1867).

$$\begin{aligned} \ddot{\theta} &= \omega^2 \sin\theta\cos\theta - (g/\ell)\sin\theta; \\ I\dot{\omega} &= T(\theta). \end{aligned}$$

- For a stable governor, we can eliminate the oscillatory degree of freedom: $\cos\theta = g/\ell\omega^2$.
- Alternatively, use implicit methods:

$$\begin{aligned} X^{j+1} &= X^j + hF(X^{j+1}) \\ X^{j+1} &= X^j + h(1 - h\mathbf{M})^{-1}F(X^j). \end{aligned}$$

Time scales (II): single particle motion



- Single particle motion in a tokamak exhibits five fundamental time-scales:
  1. The electron and ion cyclotron period

$$\tau_{cs} = m_s / e_s B$$

  2. The electron and ion transit times

$$\tau_{ts} = R / v_{ts} = (R/\rho)\tau_{cs}$$

  3. The drift time

$$\tau_D = (\rho_s / R)\omega_{ts} \sim T / eBR^2$$

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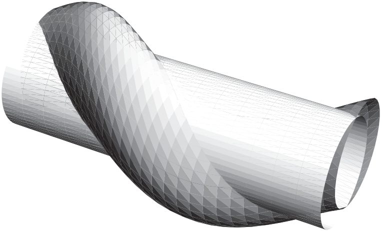
MHD assumes the plasma motion is kink-driven



- The total momentum conservation law is:

$$m_i n \frac{d\mathbf{V}}{dt} = -\nabla p - \nabla \cdot \Pi + \mathbf{J} \times \mathbf{B},$$

  where $\mathbf{V} = \mathbf{V}_i$.
- Balancing inertia against the kink force determines the time-scale

$$\tau_A = R/V_A = \beta_i^{1/2} \tau_{ti}.$$

Electron momentum conservation determines the electric field

- Using $\mathbf{V}_e = \mathbf{V} - \mathbf{J}/ne$ in the magnetic force, the electron momentum conservation takes the form

$$m_e n \frac{d\mathbf{V}_e}{dt} = ne(\mathbf{E} + \mathbf{V} \times \mathbf{B} - \eta \mathbf{J}) - \nabla p_e - \nabla \cdot \Pi_e - \mathbf{J} \times \mathbf{B}$$

- The only term that can balance the magnetic force is thus the electric force:

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0.$$

Digression: Single vs. Two-fluid models

- MHD is sometimes called a “single fluid” model. But plasma conducts electricity: $\mathbf{J} = ne(\mathbf{V}_i - \mathbf{V}_e) \neq 0$.
- It is easy to see that

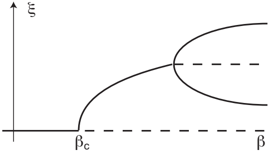
$$\frac{V_i - V_e}{V_i} \sim \frac{J}{neV_A} \sim \frac{d_i}{a} = \rho_* \beta_i^{-1/2} \ll 1,$$

where $\rho_* = \rho_i / a$.

- So MHD is a “quasi-single-fluid” in the same sense that it is a “quasi-neutral” theory:

$$\frac{n_i - n_e}{n_e} = \frac{\epsilon_0 \nabla \cdot \mathbf{E}}{n_e e} = \left(\frac{V_A}{c}\right)^2 \rho_* \beta_i^{-1/2} \ll 1.$$

Fusion MHD events are sub-Alfvénic



- Typical times scales for sawtooth crash, ELM, disruption $\sim 100\mu s \ll \tau_A \sim 1\mu s$.
- In the drinking bird toy, evaporation draws fluid up the tube, creating an inverted pendulum (c.f. D. Humphreys lecture)
- The evaporation rate $\ll \gamma$, so the dip is preceded by a precursor oscillation.
- Fusion plasmas do not fit the “drinking bird” paradigm: instabilities
  1. often lack a discernible precursor.
  2. Never satisfy ideal MHD.

Sub-Alfvenic MHD motion exhibits spatial resonances

- For $V \ll V_A$, the MHD equilibrium equation applies

$$\mathbf{J} \times \mathbf{B} = \nabla p.$$

- This determines $\mathbf{J}_\perp = (\mathbf{B} \times \nabla p)/B^2$.
- Ampere's law $\nabla \cdot \mathbf{J} = 0$ determines $J_\|$:

$$\mathbf{B} \cdot \nabla (J_\| / B) = -\nabla \cdot [(\mathbf{B} \times \nabla p)/B^2].$$

- The operator $\mathbf{B} \cdot \nabla = (m - nq)B/Rq$ is singular on magnetic surfaces with closed field lines.

Sub-Alfvénic motions are described by the drift ordering

- Recall the electron momentum conservation

$$m_e n \frac{d\mathbf{V}_e}{dt} = ne(\mathbf{E} + \mathbf{V} \times \mathbf{B} - \eta \mathbf{J}) - \nabla p_e - \nabla \cdot \Pi_e - \mathbf{J} \times \mathbf{B}$$

- Assume $\mathbf{V} \times \mathbf{B} \sim \nabla p_e / ne$ This is the *drift ordering*.
- Eliminate **E** between the electron momentum conservation and Faraday's law,

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}.$$

Hall Magnetohydrodynamics

- There follows

$$\frac{\partial \hat{\mathbf{B}}}{\partial t} = -\nabla \times \hat{\mathbf{E}},$$

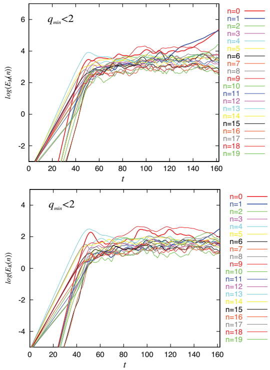
where

$$\hat{\mathbf{E}} = \mathbf{V} \times \mathbf{B} - \mathbf{J} \times \hat{\mathbf{B}}/ne - (\nabla p_e + \nabla \cdot \Pi_e)/ne;$$

$$\hat{\mathbf{B}} = (1 - d_e^2 \nabla^2)\mathbf{B};$$

$$\mathbf{J} = \nabla \times \mathbf{B}/\mu_0.$$

The perils of two-fluid models



1. Hall MHD describes whistler waves and the ion cyclotron resonance. This is the curse of the drift ordering: to describe slow evolution we have to include fast waves!

$$\omega = kV_A \quad \rightarrow \quad \omega = k^2 \rho_s V_A$$

2. By including drift motion, two-fluid models enable all the drift instabilities!
3. The model is still missing important kinetic physics (FLR, parallel dynamics, etc. . .)
4. It is hard to parallelize.

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Gyrokinetic theory offers an attractive alternative

- The drift ordering is identical to the ordering underlying the gyrokinetic theory: $k_\perp \rho_i \sim 1$ and

$$\frac{\omega}{\omega_{ci}} \sim \frac{k_\| v_{ti}}{\omega_{ci}} \sim \frac{F_1}{F_0} \sim \frac{e\phi}{T_e} \sim \frac{B_1}{B_0} \sim \rho_* \ll 1.$$

- Gyrokinetic theory is a rigorously asymptotic reduction of the Maxwell-Vlasov equations.
- It gives correct descriptions of many effects that are difficult to model using fluid moments such as
  1. Landau damping and parallel heat fluxes
  2. Finite Larmor radius
  3. Neoclassical effects

Brief outline of gyrokinetic theory

- THe GKM is based on an expansion in $\rho_* = \rho_i/a$ such that the distribution function is allowed to exhibit rapid variations across the field, but with *small amplitude* $F = F_0 + F_1$ where $F_1 = O(\rho_*)$ but $\nabla F_1 \sim \nabla F_0$.
- The equation is

$$\frac{\partial F}{\partial t} + (v_\| \mathbf{b} + \mathbf{v}_E + \mathbf{v}_D) \cdot \nabla F$$
$$\left[\frac{e}{m} E_\| - \mu\, \mathbf{b} \cdot \nabla B + v_\| (\mathbf{b} \cdot \nabla \mathbf{b}) \cdot \mathbf{v}_E\right] \frac{\partial F}{\partial v} = 0.$$

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The Gyrofluid Model (GFM): I. Moments

The gyrokinetic equations are computationally intensive. Taking their moments yields a more manageable fluid model:

$$\frac{dn}{dt} + \bar{\mathbf{b}} \cdot \nabla u_\parallel + \frac{1}{2}(\hat{\nabla}_\perp^2 \bar{\mathbf{v}}) \cdot \nabla T_\perp + \ldots = 0; \quad (1)$$

$$\frac{dP}{dt} + \bar{\mathbf{b}} \cdot \nabla p - \frac{1}{2}(\hat{\nabla}_\perp^2 \bar{\mathbf{b}}) \cdot \nabla T_\perp + \ldots = 0, \quad (2)$$

where

$$\bar{\mathbf{b}} = (\mathbf{B}_0 + \mathbf{B}_0 \times \nabla \Psi)/B_0, \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \bar{\mathbf{v}} \cdot \nabla, \qquad \bar{\mathbf{v}} = \mathbf{b}_0 \times \nabla \Phi,$$

and $\Psi = \Gamma_0^{1/2}\psi$, $\Phi = \Gamma_0^{1/2}\phi$ are the gyro-averaged magnetic and electrostatic potentials.

GFM II. Gyro-averaging

Dorland and Hammett defined the operator $\Gamma_0^{1/2}$ as follows

$$\Gamma_0^{1/2}\xi = \exp(\tfrac{1}{2}\tau\nabla_\perp^2) I_0^{1/2}(-\tau\nabla_\perp^2)\xi, \qquad (3)$$

where $\tau = T_s/T_{\mathrm{ref}}$ and I_0 is the modified Bessel function of the first kind. The definition in Eq. (3) should be interpreted in terms of its series expansion

$$\Gamma_0^{1/2}\xi = 1 + \sum_{n=1}^{\infty} a_n(\tau\nabla_\perp^2)^n = 1 + (\tau/2)\nabla^2 + \cdots ,$$

where the a_n are real numbers.

GFM III. Ampére and quasi-neutrality

- The system is completed by
  1. the quasi-neutrality equation

$$n_e = \Gamma_0^{1/2} n_i + (\Gamma_0 - 1)\phi/\tau,$$

  2. Ampére's law

$$J = \nabla^2 \psi = -\frac{\tau \beta_e}{2}(\Gamma_0^{1/2} u_i - u_e).$$

- The GFM experienced a period of success before being discredited due its overestimation of zonal flow damping.
- Its electromagnetic version is coming back into use in space physics and for studies of magnetic reconnection.

Hamiltonian versions of the GFM model can be constructed

- Why Hamiltonian? Recall that in MHD the equilibrium and charge conservation equations require that

$$\mathbf{B}\cdot\nabla(J_\parallel/B) = -\nabla\cdot(\mathbf{B}\times\nabla P/B^2).$$

- Integrability of the charge conservation condition requires that

$$\oint \frac{d\ell}{B}\nabla\cdot\mathbf{J}_\perp = 0.$$

- Similar constraints must be satisfied by the vorticity, density, etc... When are these conditions satisfied?

The integrability of the gyrofluid equilibrium is unclear

- The equilibrium equations are

$$\bar{\mathbf{v}} \cdot \nabla n + \bar{\mathbf{b}} \cdot \nabla u_{\|} + \frac{1}{2}(\hat{\nabla}_{\perp}^{2} \bar{\mathbf{v}}) \cdot \nabla T_{\perp} + \ldots = 0;$$

$$\bar{\mathbf{v}} \cdot \nabla \mathcal{P} + \bar{\mathbf{b}} \cdot \nabla p - \frac{1}{2}(\hat{\nabla}_{\perp}^{2} \bar{\mathbf{b}}) \cdot \nabla T_{\perp} + \ldots = 0,$$

$$\bar{\mathbf{v}} \cdot \nabla p_{\perp} + \bar{\mathbf{b}} \cdot \nabla u_{\|} + \frac{1}{2}(\hat{\nabla}_{\perp}^{2} \bar{\mathbf{v}}) \cdot \nabla p_{\perp} + \ldots = 0;$$

- These form a nonlinear, nonlocal system of equations to be solved for n_i, u_i, u_e, ϕ, ψ...
- Hamiltonian theory shows how to solve this seemingly intractable problem.

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Hamiltonians at work

- The ideal part of the fluid equations must be expressible in terms of Poisson brackets:

$$\partial_t \xi_j = \{\xi_j, H\} + D\nabla^2 \xi_j .$$

- Poisson brackets generally possess families of geometrical invariants C_k called Casimirs.

$$\{\xi_j, C_k\} = 0.$$

- Equilibria are the extrema of the functional $F = H + \sum_k C_k$,

$$\delta F = 0.$$

The Hamiltonian formulation thus guarantees equilibrium integrability.

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Hamiltonian formulation for Alfvén dynamics

- Consider the following model that describes kinetic and inertial-Alfvén waves:

$$\frac{\partial n_i}{\partial t} + [\Phi, n_i] = 0; \tag{4}$$

$$\frac{\partial n_e}{\partial t} + [\phi, n_e] - c_A^2 \nabla_\parallel J = 0; \tag{5}$$

$$\frac{\partial}{\partial t}(\psi - d_e^2 J) + [\phi, \psi - d_e^2 J] + \nabla_\parallel n_e = 0, \tag{6}$$

where

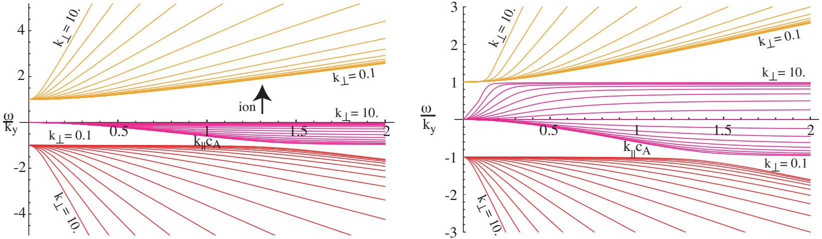
$$[f, g] = \mathbf{B}_0 \cdot (\nabla f \times \nabla g)/B_0.$$

- The conserved energy is

$$H = \tfrac{1}{2}\langle c_A^2 (|\nabla\psi|^2 + d_e^2 J^2) + n_e^2 + \Phi\, n_i - \phi\, n_e \rangle.$$

Dispersion relation

- The gyrofluid equation reproduces *exactly* the kinetic dispersion relation in both the kinetic-Alfvén and inertial regimes.
- Unlike FLR (Braginskii) models, it reproduces the band gap between the ion and electric drift frequencies.



Normal fields

- The Casimirs suggest the use of the normal fields n_i and $G_\pm = \psi - d_e^2 \nabla^2 \psi \pm \rho_e n_e$.
- The equations of motion for the normal fields are

$$\frac{\partial n_i}{\partial t} + [\Phi, n_i] = 0; \tag{7}$$

$$\frac{\partial G_\pm}{\partial t} + [\phi_\pm, G_\pm] = 0; \tag{8}$$

(9)

where

$$\phi_\pm = \phi \pm \psi/\rho_e$$

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Without magnetic reconnection, MHD instabilities would be harmless

- The “frozen-in” property of ideal MHD means that plasma cannot cross flux surfaces, and flux surfaces cannot break, so an ideally unstable plasma would either settle into a bifurcated state or bounce on the walls like a balloon.
- Magnetic reconnection prevents the saturation of MHD instabilities. The conditions for the onset of magnetic reconnection depend on plasma flows (diamagnetic and electric), FLR effects, etc…
- The onset conditions are incompletely understood.

Role of ion temperature in reconnection

- For constant-density, the model reduces to two fields:

$$\frac{\partial G_\pm}{\partial t} + [\phi_\pm, G_\pm] = 0,$$

where $G_\pm = \psi - d_e^2 \nabla^2 \psi \pm \rho_e (1 - \Gamma_0)\phi$

- Solving for ψ and ϕ yields

$$\psi = \frac{1}{2}(1 - d_e^2 \nabla^2)^{-1}(G_+ + G_-);$$
$$\phi = \frac{1}{2\rho_e}(1 - \Gamma_0)^{-1}(G_+ - G_-).$$

- We compare the cases of cold and hot ions

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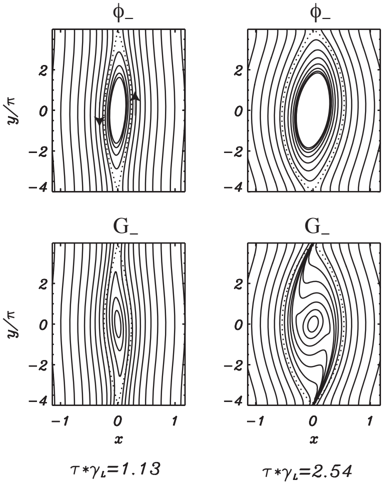
Phase-mixing in cold-ion reconnection



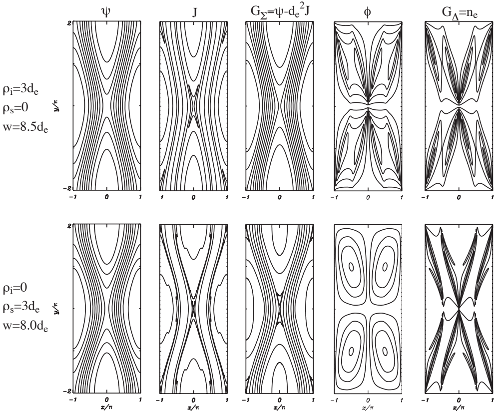
- The Lagrangian quantities $G_\pm$ are convected by the $\phi_\pm$ stream-functions.
- For cold ions,

$$(1-\Gamma_0)^{-1} = \nabla^{-2},$$

  so ϕ and ψ are smoothed.
- This is analogous to Landau damping.

(Grasso, Califano, Pegoraro and Porcelli, PRL 2001.)

Gyrofluid reconnection



- For hot ions, by contrast,

$$(1-\Gamma_0)^{-1} = -\rho_i^2 + \nabla^{-2}$$

so ϕ is not smoothed (Grasso, Califano, Pegoraro and Porcelli, Plasma Phys. Reports 2000.)

Summary (I)

- MHD is inadequate to model most MHD events in tokamaks.
- Two-fluid models provide a better description, but they still neglect important effects such as parallel heat flow, Landau damping, and finite Larmor radius.
- Two-fluid models unleash onto MHD the pandora's box of drift-acoustic turbulence: ITG, ETG, etc. . . Subgrid models may need to be developed.

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Summary (II)

- Gyrokinetic theory is free from the deficiencies of fluid models. Two of its own deficiencies are subjects of ongoing research:
  1. Quasi-static coupling to the compressional Alfvén wave;
  2. Collision operators
- The gyrofluid closure method holds considerable promise for understanding the role of FLR in MHD.
- The model that will enable us to understand ITER MHD has yet to be invented, perhaps by one of you.

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